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Duality in Parameter Space and Approximation of Measures for Mixing Repellers

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For one-dimensional expanding maps T with an invariant measure μ we consider, in a parameter space, the envelope $\mathscr{E}_n$ of the real lines associated to any couple of points of the orbit, connected by n iterations of T. If the map has s inverses and is piecewise linear, then the sets $\mathscr{E}_n$ are just the union of s^n points and converge to the invariant Cantor set of T. A correspondence between all the sets and their measures is established and allows one to associate the atomic measure on $\mathscr{E}_1$ to the çompletly continuous measure on the Cantor set. If the map is nonlinear, hyperbolic, and has s inverses, the sets $\mathscr{E}_n$ are homeomorphic to the Cantor set; they converge to the Cantor set of T and their measures converge to the measure of the Cantor set when $n \to \infty$. The correspondence between the sets $\mathscr{E}_n$ allows one to define converging approximation schemes for the map an its measure: one replaces each of the s^n disjoint sets with a point in a convenient neighborhood and a probability equal to its measure and transforms it back to the original set $\mathscr{E}_1$. All the approximations with linear Cantor systems previously proposed are recovered, the converging proprties being straightforward in the present scheme. Moreover, extensions to higher dimensionality and to nondisconnected repellers arte possible and are briefly examined.

KEY WORDS: Mixing repellers; Cantor set; Julia set; linear approximation; p-balanced measures.

1. INTRODUCTION

Among the nontrivial dynamical systems whose parameters specifying the ergodic and geometric properties can be explicitly evaluated,[1,2,4] certainly the linear Cantors aree the most relevant. As a consequence, it is nature to

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approximate the nonlinear Cantors such as the mixing repellers[3] with sequences of linear Cantors. A former approximation scheme was based[4-6] on sequences of linear Cantors whose first-order partition agrees with the order-n partition for the nonlinear Cantor. Convergence results for the free energy were obtained, not for the measures.

The analysis of the convergence properties of the measures of the approximating sequence and the classification of all possible approximation schemes remained to be done.

These problems are solved in the present work, using a duality between the configuration space and a *parameter space* and a renormalization scheme. The basic idea is the following: given an expanding map T on the real line, whose repeller is a Cantor set C with an invariant measure μ, we consider the envelope $\mathscr{E}_n$ of the straight lines $\mathscr{L}_n$ in a parameter space (λ, b) defined by $x_{n+k} = \lambda x_k + b$, where x_k is any point of the orbit and x_{n+k} is obtained by n backward iterations $T^n(x_{n+k}) = x_k$. The measure μ on the Cantor induces a measure μ_n on $\mathscr{E}_n$. When $n \to \infty$ the sequence $\mathscr{E}_n$ converges to C, while μ_n converge to μ on C. For a linear Cantor, $\mathscr{E}_n$ is the union of s^n distinct points and the measure μ_n is atomic. Any set $\mathscr{E}_n$ of the sequence can be mapped back to $\mathscr{E} \equiv \mathscr{E}_1$. As a consequence, one can associate to all the atomic measures μ_n on $\mathscr{E}_n$ and the completely continuous measure on the Cantor to which they converge the atomic measures with s masses on $\mathscr{E}$.

When the Cantor is nonlinear (since T is nonlinear), $\mathscr{E}_n$ are given by union of s^n sets $\mathscr{E}_{n,j}$ homeomorphic to the Cantor C. In this case we replace each of the sets $\mathscr{E}_{n,j}$ with a point in a suitable neighborhood, associate to it a probability p_j equal to its measure, and transform these points back to $\mathscr{E}$, obtaining a discrete set $\mathscr{L}^{(n)}$, with the corresponding atomic measure. This corresponds to a linear Cantor system, in ordinary space, with s^n scales, but in general overlapping preimages. The convergence $\mathscr{L}^{(n)}$ to $\mathscr{E}$ and of the corresponding measures together with the correspondence between the Cantor C itself and $\mathscr{E}$ allows us to establish the approximation theory with linear Cantor systems.

This result is significant not only because it induces a duality between the configuration and parameter space, but also because it allows us to define in a rigorous way a sequence of simple converging approximations of the repeller and its measure. The envelope method in the parameter space allows us to extend the linear approximation method also to connected repellers such as the quadratic Julia sets of a quadratic map $z' = z^2 - p$ for p close to 0. The case of the circle $p = 0$ will be explicitly discussed. The method seems to fail, at least as far as uniform convergence is concerned, when the system is no longer hyperbolic, as is the case for the Julia set with $p = 2$. Extensions to higher dimensions are rather straightforward.

The plan of the paper is as follows: in Section 2 we consider the case of linear maps and show that there is a natural correspondence between points lying on the Cantor set and the p-balanced measure supported on it on one side and the lines which produce the envelope and the measure induced on them on the other; the limiting envelope is the Cantor set itself. In Section 3 we discuss the nonlinear case and state that each kth-level envelope and its measure are homeomorphic to the Cantor set and its measure; again the limiting envelope is the Cantor set itself. Section 4 deals with the particular case of quadratic maps; we introduce the renormalization transform on the envelopes and give convergence results to the atractor and the p-balanced measure associated to it. It should be noticed that this result does not depend on the form of the map and so can be immediately generalized to any finite set of nonlinear maps T_k which satisfy condition (2.1) below. In Section 5 we generalize the approximation and renormalization scheme to sequences of points which lay out of the envelopes such as those in refs. 4–6. Section 6 deals with the nondisconnected cases $p = 0, 2$, while in the Appendix some remarks are given for the case in which the maps do not depend only on the first predecessor element of the sequence, but the first envelope is "degenerate."

2. LINEAR CANTOR SYSTEMS

We consider a mixing repeller (T, μ, C) where C is a Cantor set and μ is an invariant measure on C. We assume $C \subset I \equiv [0, 1]$, $\operatorname{diam}(C) = 1$,

$$T^{-1}(I) = \bigcup_{j=1}^{s} I_k, \qquad I_k \cap I_j = \varnothing, \quad k \neq j, \quad k, j = 1,..., s \tag{2.1}$$

and that T is expanding

$$|T(x) - T(y)| \geqslant a|x - y|, \qquad a > 1 \tag{2.2}$$

The inverses of T on I_j will be denoted by T_j rather than T_j^{-1} to simplify the notation and we shall denote the elements of the order-n partition of C by

$$I_{j_1,...,j_n} = T_{j_n} \circ \cdots \circ T_{j_1}(I), \qquad 1 \leqslant j_k \leqslant s, \quad 1 \leqslant k \leqslant n \tag{2.3}$$

The measure μ is specified by the probabilities $p_1,..., p_s$ associated with $T_1,..., T_s$, respectively. The time series or orbit $\bar{x} = \{x_0, x_1,..., x_k,...\}$, where $x_0 \in C$ and $x_{k+1} = T_j(x_k)$ with probability p_j, is dense on C except for a set of measure zero of initial conditions (fixed points). This system of inverse maps and their probabilities is known also as an iterated function system (IFS).[7–10]

The equations of the straight lines in the parameter space b, λ are

$$x_{n+k} = \lambda x_k + b, \qquad k = 1, 2,... \tag{2.4}$$

where $T^{\circ n}(x_n + k) = x_k$. In a more explicit notation, given an $x \equiv x_k$, any of its s^n preimages of order n should be labeled with a set of n indices and we will write

$$x_{j_1,..., j_n} \equiv T_{j_n} \circ \cdots \circ T_{j_1}(x) = \lambda x + b, \qquad x \in [0, 1] \tag{2.5}$$

The envelope of these straight lines is given by

$$\begin{aligned} \lambda &= \frac{d}{dx} x_{j_1,..., j_n}(x) \\ b &= x_{j_1,..., j_n}(x) - \lambda x \end{aligned} \tag{2.6}$$

When $x \in I$, Eq. (2.6) defines s^n disjoint curves in the plane b, λ which we shall denote by $\lambda^{(n)}_{j_1,..., j_n}(x)$, $b^{(n)}_{j_1,..., j_n}(x)$. If $x \in C$, as is the case for $x \equiv x_k$ a point of the orbit, then each curve is replaced by a set homeomorphic to the Cantor. To any arc $\lambda^{(n)}_{j_1,..., j_n}(x)$, $b^{(n)}_{j_1,..., j_n}(x)$ we associate the measure $\nu(\lambda, b) = \mu(x_{j_1,..., j_n}(x))$, so that the measure of the full arc is $p_{j_1} \cdots p_{j_n}$.

We shall first examine the trivial case of a linear Cantor system,

$$T_k(x) \equiv L_k(x) = \lambda_k x + b_k, \qquad k = 1,..., s \tag{2.7}$$

It is evident that lines (2.4) in the λ, b plane pass through the point $\{\lambda_k, b_k\}$ if $x_n = L_k(x_{n-1})$. Given a pair (x_{n-1}, x_n), it identifies in a unique manner one of the possible pairs $\{\lambda_k, b_k\}$; here the nonoverlapping condition (2.1) is essential to avoid ambiguities in the association. In the linear case the Legendre transform produces a singular envelope given by s points $\{\lambda_k, b_k\}$, $k = 1,..., s$.

The structure of a single bunch of lines, of the intersections of all the lines of one bunch with a single line of a second one, and, finally, the set of intersections of the lines of two bunches is in a natural correspondence with the structure of C:

Proposition 1. (i) The closure of the set of translation coefficients of the kth arc is diffeomorphic to C. The same is true for the angular coefficients of the lines. The measure associated to these sets is diffeomorphic to the p-balanced measure of C restricted to $L_k(I)$.

(ii) The closure of the set of intersections of the lines of a bunch with a fixed line of a second one is diffeomorphic to C. The measure associated to this set is diffeomorphic to the p-balanced measure of C.

(iii) The closure of the set of intersections of the lines of the kth bunch with those of the ith one is diffeomorphic to $C \times C$. The measure associated is $p_k p_i \mu \times \mu$.

(iv) Statements (i)–(iii) are still true in the case in which $p_i = p_i(x)$ are the probability functions associated to an hyperbolic IFS which satisfies (2.1) and (2.2) and p_i: $I \to [0, 1]$ are continuous functions, uniformly bounded away from zero, whose continuity modules satisfy Dini's condition[11–13] and moreover $\sum p_i(x) = 1$ for any $x \in I$.

The proof follows trivially from the nonoverlapping hypothesis and the density of the series. From now on, if not differently stated, it will be assumed that p_i are constant functions. In the case of the linear maps the Legendre transform produces a noticeable simplification of the structure of the Cantor set and gives the parameters of the IFS in a natural way. The λ_i and b_i are simply the centers of the bunches of lines, and the probabilities p_i become the weights of the Dirac measure associated to $\{\lambda_i, b_i\}$. In fact the following result is true.

Proposition 2. The measure associated to the envelope generated by the Legendre transform of the maps is

$$dv(\lambda, b) = \sum_{i=1}^{s} p_i\, \delta(\lambda - \lambda_i)\, \delta(b - b_i)\, d\lambda\, db \tag{2.8}$$

The proof is trivial and is omitted.

Propositions 1 and 2 are still true, with obvious changes in notations, if we consider the iterate of order n. That corresponds to considering the lines generated by pairs $\{x_k, x_{k+n}\}$; those envelopes give s^n points $c_{\mathbf{i}}^{(n)} \equiv \{\lambda_{\mathbf{i}}^{(n)}, b_{\mathbf{i}}^{(n)}\}$, where $\mathbf{i} = i_1,..., i_n$ is a multi-index.

The following result is true:

Proposition 3. We have

$$\text{Closure}\left(\lim_{n \to +\infty} \left\{\bigcup_{\mathbf{i}} c_{\mathbf{i}}^{(n)}\right\}\right) = C \tag{2.9}$$

Proof. From the hyperbolicity hypothesis of the IFS it follows that

$$\begin{aligned} &\lim_{n \to +\infty} \lambda^{(n)} = 0, \qquad \lambda^{(n)} = \max_{\mathbf{j}} \lambda_{\mathbf{j}}^{(n)} \\ &\lim_{n \to \infty} b_{\mathbf{j}}^{(n)}(\mathbf{x}) = x \end{aligned} \tag{2.10}$$

and, since the series is dense in C, then $d_H(\{b_j^{(k)}\}, C) \leqslant \lambda^{(k)}$. So that the assertion easily follows.

If we allow overlapping of the images of maps, the correspondence between the pair $\{x_{n+k}, x_n\}$ and $c_i^{(n)}$ fails to be one-to-one for some n whatever k is: it happens that the line corresponding to such pairs passes through more than one center $c_i^{(n)}$. Also in this case we expect that the limit envelope is C, but there are problems in associating a measure to the nth envelope; this ambiguity is reflected in the fact that in this case the p-balanced measure associated to such Cantor sets is not completely characterized depending on parameter space, not even in the case of two maps.[8]

3. NONLINEAR CANTOR SETS

Let us consider the case of nonlinear maps T_i, $i = 1,..., s$, with constant probabilities p_i. Let us consider first one single piece of the envelope

$$b^{(1)}(x) = -\lambda^{(1)}(x)x + T(x) \tag{3.1}$$

where $x \in \bar{x}$, $T = T_k$, or some $k = 1,..., s$. We then get the following parametric expressions for $\lambda^{(1)}$, $b^{(1)}$:

$$\begin{aligned} \lambda^{(1)}(x) &= T'(x) \\ b^{(1)}(x) &= -T'(x)x + T(x) \end{aligned} \tag{3.2a}$$

if we pass to the second iterate of the maps, analogously we get

$$\begin{aligned} \lambda_{i,j}^{(2)}(x) &= T_j'(T_i x)\, T_i'(x) \\ &= \lambda_j^{(1)}(T_i x)\, \lambda_i^{(1)}(x) \\ &= \lambda_j^{(1)}(T_i x)\, \lambda_i^{(1)}(T(T_i(x)) \\ b_{ij}^{(2)}(x) &= -T_j'(T_i(x)x + T_j(T_i x) \\ &= -\lambda_j^{(1)}(T_i x)\, \lambda_i^{(1)}(x)x + T_j(T_i(x)) \\ &= \lambda_j^{(1)}(T_i x)\, b_i^{(1)}(x) + b_j^{(1)}(T_i x) \\ &= \lambda_j^{(1)}(T_i x)\, b_i^{(1)}(T(T_i(x)) + b_j^{(1)}(T_i x) \end{aligned} \tag{3.2b}$$

Note the analogy of (3.2b) with the expression for the parameters of the second iterate of a linear IFS:

$$\begin{aligned} \lambda_{ij} &= \lambda_j \lambda_i \\ b_{ij} &= b_j \lambda_i + b_i \end{aligned}$$

(3.2b) generalizes to the case of the nth iterate of the IFS in the following way:

$$\begin{aligned}
\lambda^{(n)}_{i_1,\dots,i_n}(x) &= \lambda^{(1)}_{i_n}(x_{i_1,\dots,i_{n-1}}(x))\cdots\lambda^{(1)}_{i_n}(x)\\
&= \lambda^{(1)}_{i_n}(x_{i_1,\dots,i_{n-1}}(x))\cdots\lambda^{(1)}_{i_1}(T^{\circ(n-1)}(x_{i_1,\dots,i_{n-1}}(x))\\
b^{(n)}_{i_1,\dots,i_n}(x) &= -\prod_{k=1}^{n}\lambda_{i_k}(x_{i_1,\dots,i_{k-1}}(x))x + x_{i_1,\dots,i_n}(x) \qquad (3.2c)\\
&= \lambda^{(n-1)}_{i_1,\dots,i_{n-1}}(T_{i_n}x)\, b^{(1)}_{i_n}(x) + b^{(n-1)}_{i_1,\dots,i_{n-1}}(T_{i_n}x)\\
&= \lambda^{(n-1)}_{i_1,\dots,i_{n-1}}(T_{i_n}x)\, b^{(1)}_{i_n}(T(T_{i_n}(x)) + b^{(n-1)}_{i_1,\dots,i_{n-1}}(T_{i_n}x)
\end{aligned}$$

where $x_{j_1,\dots,j_n} = T_{j_n}\circ\cdots\circ T_{j_1}(x)$ and the analogy with the linear case still holds true.

In the case of the inverse determinations of the quadratic maps $T_\pm = \pm(x+p)^{1/2}$, the recursion relations for $b^{(n)}$ can be written also in another way, due to the fact that

$$(T_\pm)'(x) = \pm\frac{1}{2T_\pm(x)}$$

In fact it easy to check that

$$\begin{aligned}
b^{(l)}_{i_1,\dots,i_n} &= p\lambda^{(1)}_{i_1}(x_{i_2,\dots,i_n}(x))\left[\frac{1}{2^{k-1}} + \lambda^{(1)}_{i_2}(x_{i_3,\dots,i_n}(x))\left[\cdots\left[\frac{1}{2}+\lambda^{(1)}_{i_n}x)\right]\cdots\right]\right]\\
&= p\lambda^{(l)}_{i_1,\dots,i_n}x) + \frac{1}{2}b^{(n-1)}_{i_2,\dots,i_n}(x)
\end{aligned}$$

In the linear case the envelope is degenerate and reduces to a finite number of points. In the strictly nonlinear case this is not possible if the maps T_i are not linear in any interval. The envelope is a set of curves $C_{i_1\cdots i_n}$, one for each map, and is parametrized by x. If we take the admissible points which stay in the sequence $\bar{x}$, we get the following result.

Proposition 4. The admissible points of each algebraic curve $C_{i_1\cdots i_n}$ form a set $\tilde{C}_{i_1\cdots i_n}$ which is homeomorphic to the corresponding image of the Canter set $T_{i_n}\circ\cdots\circ T_{i_1}(C)$. The measure $\tilde{\mu}_{\tilde{C}_{i_1\cdots i_n}}$ supported by the set is homeomorphic to the p-balanced measure μ restricted to $T_{i_n}\circ\cdots\circ T_{i_1}(C)$.

It should be noticed that the homeomorphisms is not a diffeomorphism in general, since we do not make any special hypothesis of regularity for the derivatives of the maps T_i.

Proposition 3 can be generalized so as to hold also in the nonlinear case, and we get the following result.

Proposition 5. We have

$$\lim_{n \to +\infty} \max_{i_1,\ldots,i_n} \sup_{x \in \bar{x}} |\lambda^{(n)}_{i_1,\ldots,i_n}(x)| = 0$$
$$b^{\infty} = \lim_{n \to +\infty} \sup_{x \in \bar{x}} b^{(n)}_{i_1,\ldots,i_n}(x) = x_{i_1,\ldots,i_n,\ldots} \in C \tag{3.3}$$

where the limit of $b_{i_1,\ldots,i_l}$ in (3.3) is uniform in $x \in \bar{x}$. The closure of the limit envelope is the Cantor set C itself.

The proofs are trivial and are omitted.

The generalization to the multidimensional case is straightforward and all the results on unidimensional linear maps stated in Propositions 1–3 can be easily generalized to this case. More generally, we consider a map T with s inverses defined on $T^{-1}([0, 1]^d)$ which we assume to be the union of s disconnected sets I_k. When the inverse maps $L_k(\mathbf{x})$ are linear and contracting on I_k the limit invariant set is a Cantor. We shall write $L_k(\mathbf{x}) = \mathbf{L}_k\mathbf{x} + \mathbf{b}_k$, where $\mathbf{L}_k$ is a $d \times d$ matrix and $\mathbf{b}_k$ a vector of $\mathbf{R}^d$, so that any linear map depends on $d(d+1)$ parameters. Given any couple of points $\mathbf{x}_k$, $\mathbf{x}_{n+k}$ of an orbit of a linear or nonlinear Cantor such that $\mathbf{x}_k = T^{\circ n}(\mathbf{x}_{n+k})$ in the $\mathbf{R}^{d(d+1)}$ parameter space, we consider the d^2-dimensional linear manifold

$$\mathbf{x}_{n+k} = \mathbf{L}\mathbf{x}_k + \mathbf{b}_k \tag{3.4}$$

vector. If we let $\mathbf{x}_{j_1,\ldots,j_n} = T_{j_n} \circ \cdots \circ T_{j_1}(\mathbf{x})$ be the nth preimage of a point, we shall write

$$\mathbf{x}_{j_1,\ldots,j_n} = \mathbf{L}\mathbf{x}_k + \mathbf{b}_k \tag{3.5}$$

and obtain the d^2-dimensional manifolds in parameter space as the envelope of the above linear manifold, namely

$$\mathbf{L}_{i,l} = \frac{\partial}{\partial x_{(l)}} (x_{(i)})_{j_1,\ldots,j_n}(x)$$
$$b_i = \mathbf{L}_{i,l}\, x_{(l)} - (x_{(i)})_{j_1,\ldots,j_n} \tag{3.6}$$

where we denote by $x_{(l)}$ the lth component of the vector $\mathbf{x}$.

4. MEASURE CONSTRUCTION FOR HYPERBOLIC JULIA SETS

We consider the explicit case of a quadratic map $T(x) = x^2 - p$ for $p > 2$. The inverse maps are then denoted by $T_{\pm}$ and read

$$T_{\pm}(x) = \pm(x+p)^{1/2} = \eta(x+p)^{1/2} \tag{4.1}$$

and their probabilities are denoted by $p_{\pm}$. The Julia set C is contained in the interval $[-q, q]$, where $q=[1+(4p+1)^{1/2}]/2$. In this case the envelope (3.1) can be expressed explicitly as a function of

$$\lambda^{(1)}(x)=T'(x)=\frac{\eta}{2}\frac{1}{(x+p)^{1/2}}=\frac{\eta}{2}\frac{1}{T(x)}$$

so that

$$b^{(1)}=\frac{1}{4\lambda^{(1)}}+p\lambda^{(1)} \tag{4.2}$$

and the statements of Propositions 4 and 5 and of Corollary 1 hold true. In Fig. 1, we show the first envelopes for the case $p=3$, where $x\in[-q, q]$; note the self-similarity between the first-order envelope and subsequent ones which correspond to the arcs $(j_1,..., j_n)$ with $j_i=j_k$ for all $i, k=1,..., n$.

We want to use these envelopes to show that, under certain natural constraints, the limit of the attractors and p-balanced measures of a sequence of linear IFSs is the Julia set $J\equiv C$ nd its measure. As a particular case, it follows that it is possible to reconstruct the measure of the Julia set with the sequence of linear Cantor sets introduced in refs. 4–6. The results stated here be generalized, for instance, to the case of nonquadratic maps and also in dimension greater than one without any problem.

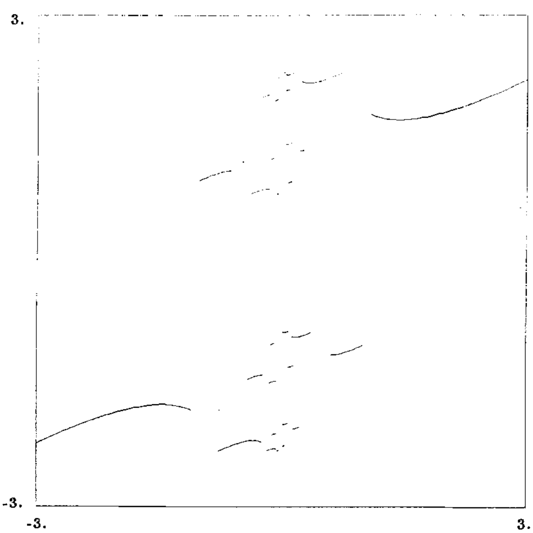

Fig. 1. The first four envelopes for the case of the quadratic map $T=x^2-3$.

Let us start with the nth-order envelope $\mathscr{E}_n$ in the parameter space. We fix $x=\bar{x}$ and the corresponding point on each of the s^n arcs of the envelope. We can then renormalize these s^n points back to the first-order envelope $\mathscr{E}$. To the point on the arc $(j_1,..., j_n)$ we associate a probability equal to the measure of the arc $p^{(n)}_{j_1 \cdots j_n} = \mu(T_{j_1} \circ \cdots \circ T_{j_n}(J))$. One could equally well choose arbitrarily the points on each arc, namely on the arc $(j_1,..., j_n)$ the point corresponding to $x = \tilde{x}^{(j_1,...,j_n)} \in [0, 1]$. We can then renormalize these points to the first envelope using expression (3.2c) and associating to $(\lambda^{(n)}_{j_n \cdots j_1}(\tilde{x}),\ b^{(n)}_{j_n \cdots j_1}(\tilde{x}))$ the point $(\lambda^{(1)}_{j_1}(x_{j_n \cdots j_2}(\tilde{x})),$ $b^{(1)}_{j_1}(x_{j_n \cdots j_2}(\tilde{x})))$, where $\tilde{x}$ can be replaced by a different point for any arc, namely by $\tilde{x}^{(j_1,...,j_n)} \in [0, 1]$.

In this way we put on the first envelope s^n points each with a defined probability equal to the measure of the corresponding element on the order-n partition of the Cantor set. If we let $n \to +\infty$, the sequence on points on the first envelope becomes dense on the image of the Cantor in $\mathscr{E}$. The corresponding measure becomes the measure v on $\mathscr{E}$ corresponding to the measure μ on C.

It is relevant to illustrate the meaning of the limiting procedure and the renormalization scheme. We first observe that the set $\mathscr{E}_n$ and the measure $v_n(\lambda, b)$ on it correspond to the dynamical system $(T^{\circ n}, \mu, C)$, while $\mathscr{E}$ and its measure v correspond to (T, μ, C). As a consequence, when we replace $\mathscr{E}_n$ and v_n by a set of s^n on each arc and their probabilities equal to the measure of the arcs we construct a linear Cantor system (L_n, μ_n, C_n) such that $\mu_n \to \mu$ and $C_n \to C$ when $n \to \infty$. There is a problem in computing the dynamical variables for L_n in this limit. However, since the free energy $\mathscr{F}$[14] scales with n, namely $\mathscr{F}(T^{\circ n}, \mu) = n\mathscr{F}(T, \mu)$, one can expect that the limit of $\mathscr{F}(L_n, \mu_n)/n$ exists and is equal to $\mathscr{F}(T, \mu)$. The renormalization procedure removes this difficulty, since it produces a sequence of renormalized linear Cantor systems $(\hat{L}_n, \hat{\mu}_n, \hat{C}_n)$ converging to (T, μ, C). Any dynamical or thermodynamic variable for the (T, μ, C) is given by the limit of the corresponding dynamical and thermodynamic variables on the sequence of linear Cantor systems. We have therefore two families of approximation schemes. The only disadvantage of the renormalized scheme is that it produces linear Cantor systems with overlapping preimages.

Theorem 1. Given a sequence of points

$$(\lambda^{(n)}_{i_n \cdots i_1}(\tilde{x}), b^{(n)}_{i_n \cdots i_1}(\tilde{x})), \qquad i_j = +, -, \quad j = 1,..., n$$

each on one of the s^n elements of the nth envelope with associated probability $p^{(n)}_{i_n \cdots i_1} = \mu(T_{i_1 \cdots i_n}(J))$, and renormalizing them to the first envelope

$\tilde{C}_1$ via the association to each point of $(\lambda_{i_1}^{(1)}(x_{i_n \cdots i_2}(\tilde{x})),\ b_{i_1}^{(1)}(x_{i_n \cdots i_2}(\tilde{x})))$, then

$$\text{Closure}\left\{\lim_{n \to +\infty} \bigcup_{i_1,\ldots,i_n} (\lambda_{i_1}^{(1)}(x_{i_n \cdots i_2}(\tilde{x})),\ b_{i_1}^{(1)}(x_{i_n \cdots i_2}(\tilde{x})))\right\} = \tilde{C}_{i_1}$$

and the measure associated to the closure of the limiting set is $\mu_{|\tilde{C}_{i_1}}$.

It should be noticed that the ordering of the indices in all of the formulas is essential throughout all of this section.

In the case $x = q$ it is possible to carry out explicitly the calculations of renormalization. It should be noticed that, once the renormalized λ's are obtained, the renormalized b's are obtained in a straightforward manner from the recurrence formulas (3.2c). The calculations of the set of renormalized λ's can be done in a hierarchical way, starting from $\lambda_{+\cdots+}^{(l)}(q) = (\lambda_{+}^{(1)})^{l}(q)$ and then, once the renormalized parameter corresponding to a certain sequence of $+$ and $-$ of length l, $\{i_l, i_{l-1},\ldots, i_1\}$, is found, at the next stage one obtains those corresponding to $\{i_{l-1},\ldots, i_1, i_l\}$ and $\{i_{l-1},\ldots, i_1, -i_l\}$. In this way it can be easily checked that all the 2^l points of the sequence are renormalized.

The renormalization scheme can also be easily carried out if we choose $\tilde{x}$ different for the various $\lambda_{j_1,\ldots,j_n}(\tilde{x})$, $b_{j_1,\ldots,j_n}(\tilde{x})$. This happens if $\tilde{x} = \tilde{x}^{(j_1,\ldots,j_{n-1})}$ are chosen to be the fixed points of $T_{j_1} \circ \cdots \circ T_{j_{n-1}}$, namely

$$x_{j_1,\ldots,j_{n-1}}(\tilde{x}^{(j_1,\ldots,j_{n-1})}) = \tilde{x}^{(j_1,\ldots,j_{n-1})}$$

5. APPROXIMATION SCHEMES

We show that even when the points are not in the arcs of the envelope $\mathscr{E}_n$ the renormalization scheme can be carried out. Let $(\tilde{\lambda}_{i_n \cdots i_1}^{(n)},\ \tilde{b}_{i_n \cdots i_1}^{(n)})$ be the points chosen in a neighborhood of each arc of the envelope $\mathscr{E}_n$. If we choose the neighborhood to be a disc containing the arc and with radius equal to the diameter of the arc or the convex hull of the arc, when $n \to \infty$ the limit obtained by choosing any point on the arc or on its neighborhood is the same. The major difficulty in this case is to associate to each point of one element of the sequence a point in the plane of the first envelope so that as $n \to +\infty$, they approach the right points on it. The problem comes from the fact that only when the points of the sequence stay on the envelope it is possible to use expression (3.2c) in order to renormalize them to the first envelope.

Here we propose two "natural" ways of "renormalizing" them and we expect that any reasonable method should give the same limit result if the points of the sequence satisfy some natural constraint.

Let us suppose that the points of the sequence stay inside the convex hull of the corresponding element of the envelope, which means that

$$\lambda^{(n)}_{i_n \cdots i_1 \min} \leqslant \tilde{\lambda}^{(n)}_{i_n \cdots i_1} \leqslant \lambda^{(n)}_{i_n \cdots i_1 \max}$$
$$b^{(n)}_{i_n \cdots i_1 \min} \leqslant \tilde{b}^{(n)}_{i_n \cdots i_1} \leqslant b^{(n)}_{i_n \cdots i_1 \max}$$

where min and max refer obviously to the minimal and maximal λ, b in the $i_n \cdots i_1$th part of the nth envelope. In this case from Proposition 5 it follows easily that $\tilde{\lambda}^{(n)}_{i_n \cdots i_1}$, $\tilde{b}^{(n)}_{i_n \cdots i_1}$ go to the expected limiting points in the limit of infinite n. In order to renormalize to the first envelope, notice that there exists a unique $\tilde{x} \in [-q, q]$ such that $\{\tilde{\lambda}^{(n)}_{i_n \cdots i_1} = \lambda^{(n)}_{i_n \cdots i_1} \tilde{x}\}$ and there is at least one point and there are at most two $x_1, x_2 \in [-q, q]$ such that the same is true for $\tilde{b}^{(n)}_{i_n \cdots i_1}$. So projecting each point on the λ and b axes, we get essentially one or two points inside the convex hull of the first envelope in the part we have denoted previously as the $i_n \cdots i_1$th. It is clear that in the limit we get convergence also in this case, since due to the uniformity of the limits in Proposition 5, there is no serious problem coming from the lack of unicity of the "renormalized point."

If the points stay outside the convex hull, but approach the envelope for $n \to +\infty$ and the scales go uniformly to zero as $n \to +\infty$, then one can still project $\{\tilde{\lambda}^{(n)}_{i_n \cdots i_1}, \tilde{b}^{(n)}_{i_n \cdots i_1}\}$ on the λ axis in a unique way, but it is impossible to associate by another orthogonal projection any b in the allowed set given by the envelope. In this case the idea is to take the two tangents to the envelope which come out of the point itself. If for n sufficiently big they touch allowed points of the envelope, we are done and we can associate to each of these point a renormalized one which stays outside of the first envelope and is given by the intersection of the corresponding renormalized tangents to the first envelope. In this way we still get a point in the $i_n \cdots i_1$th part of the first envelope as desired and we proceed as before.

The discussion above is the basical argument in the proof of the following result.

Theorem 2. Let a sequence of points

$$\tilde{\lambda}^{(n)}_{i_n \cdots i_1}, \tilde{b}^{(n)}_{i_n \cdots i_1}), \qquad i_j = +, -, \quad j = 1, \ldots, N$$

each one inside the convex hull generated by the corresponding $i_n \cdots i_1$th element of the partition of order n be given. Suppose that to each element of the sequence is associated a probability $\tilde{p}^{(n)}_{i_n \cdots i_1} = \mu(T_{i_1} \circ \cdots \circ T_{i_n} J))$. Renormalize these points to the convex hull of the first envelope $\tilde{C}_{i_1}$ according to the method proposed above, associating to each one the corresponding

$$\{\lambda^{(1)}_{i_1}(x_{i_n, \ldots, i_2}(\tilde{x})), b^{(1)}_{i_1}(x_{i_n, \ldots, i_2}(\tilde{x}'))\}$$

where $\tilde{x}'$ is either x_1 or x_2 in the notations above.

Then

$$\text{Closure}\left\{\lim_{n\to+\infty}\bigcup_{i_1,\ldots,i_n}\tilde{\lambda}_{i_1}^{(1)}(x_{i_n\cdots i_2}(\tilde{x})),\, b_{i_1}^{(1)}(x_{i_n\cdots i_2}(\tilde{x}')))\right\}=\tilde{C}_{i_1}$$

and the measure associated to the closure of the limiting set is $\mu_{|\tilde{C}_{i_1}}$.

In the case the points lay outside of the hull, but are such that the two tangents coming out of them touch the corresponding element of the partition in points which are the Legendre transform of points $\hat{x}^1$, $\hat{x}^2$ in $T_{i_1}\circ\cdots\circ T_{i_n}[-q,q]$, associate the intersections of the tangents to the envelope coming out of the renormalized points

$$\{\lambda_{i_1}^{(1)}(x_{i_n,\ldots,i_2}(\hat{x}^1)),\, b_{i_1}^{(1)}(x_{i_n,\ldots,i_2}(\hat{x}^1))\}$$
$$\{\lambda_{i_1}^{(1)}(x_{i_n,\ldots,i_2}(\hat{x}^2)),\, b_{i_1}^{(1)}(x_{i_n,\ldots,i_2}(\hat{x}^2))\}$$

Let us associate probabilities to each point as before. Then, also in this case, the closure of the limit set is $\tilde{C}_{i_1}$ and the measure associated to the closure of the limit set is $\mu_{|\tilde{C}_{i_1}}$.

In the case considered by Turchetti and Vaienti,[4–6] the

$$\tilde{\lambda}_{i_n\cdots i_1}^{(n)}=(2q)^{-1}\operatorname{diam}(T_{i_1}\circ\cdots\circ T_{i_n}(J))$$

where diam denotes the diameter, and are such that

$$\lambda_{i_n\cdots i_1\min}^{(n)}\leqslant\tilde{\lambda}_{i_n\cdots i_1}^{(n)}\leqslant\lambda_{i_n\cdots i_1\max}^{(n)}$$

The $\tilde{b}_{i_n\cdots i_1}^{(n)}$ are chosen in such a way that the image of the linear map is $L_{i_n\cdots i_1}^{(n)}([-q,q])=T_{i_n\cdots i_1}([-q,q])$ and it can be easily verified that the hypotheses of Theorem 2, part 2 are verified in this case. The points of the sequence are given by the intersection of the tangents coming out of the Legendre transform of the iterated extremal points $-q$, q of the Julia set J.

Theorems 1 and 2 can be straighforwardly implemented in the cases of any finite set of nonlinear maps T_k, $k=1,\ldots,N$, which satisfy (2.1) and (2.2) and in higher dimensions than one.

One can interpret the regions in which one gets convergence to the measure and the attractor of the target Cantor set, by Theorem 2, as a subset of the basin of attraction of the first envelope associated to the Cantor set itself.

6. SOME REMARKS ON THE NON-CANTORIAN CASES $p=0$ AND $p=2$

In this section we consider two particular cases which correspond to nonhyperbolic Julia sets, namely $p=0$ and $p=2$. In case $p=0$, Proposition 5 can be modified to hold, while it seems to fail in the case $p=2$, due to the fact that the critical point $z=0$ stays on the Julia set J. In this last case, since the critical point $z=0$ lies on the Julia set J, the limit envelope fails to be in a natural correspondence with the Julia set and one can recover a homeomorphism between the first envelope and the Julia set going to the one-point compactification of $\mathbb{C}$. What happens in these two cases shows that some of the results stated throughout this paper do not depend on the hyperbolic structure of the Julia set and suggests the possibility of investigating more general sets of maps.

For simplicity, in the following we will change our notation and make calculations using the direct map $T(z)=z^2-p$ instead of its two inverse branches $T_{\pm}(x)=\pm(x+p)^{1/2}$. This does not affect the results stated in the previous sections, of course.

In the case $p=0$, the map is $T(z)=z^2$, where $z\in\mathbb{C}$ and the Julia set J is simply the unit circle in the complex plane $J=\{z\in\mathbb{C}\colon |z|=1\}$. Using the direct map, Eq. (3.1) becomes

$$b^{(1)}=-\lambda^{(1)}T(z)+z \tag{6.1}$$

and parametrically we get

$$\begin{aligned} \lambda^{(1)}(z)&=\frac{1}{T'(z)}=\frac{1}{2z} \\ b^{(1)}(z)&=-\frac{T(z)}{T'(z)}+z=\frac{z}{2} \end{aligned} \tag{6.2a}$$

So the envelope is a closed curve which lies in the complex plane $\{\lambda, b\}$, as expected. Since we are interested in the points of the form $z=e^{i\theta}$, $\theta\in[0, 2\pi)$, by the coordinate transform

$$\lambda^{(1)}=\tfrac{1}{2}\exp iA^{(1)}$$

$$b^{(1)}=\tfrac{1}{2}\exp iB^{(1)}$$

we get parametrically

$$A^{(1)}(\theta)=-\theta$$

$$B^{(1)}(\theta)=\theta$$

where $\theta\in[0, 2\pi)$ or $B^{(1)}=-A^{(1)}$.

If we consider the kth iterate of the map T, $T^{(k)}(z)=z^{2k}$, then we get

$$\begin{aligned} \lambda^{(k)}(z)&=\frac{1}{2^k}z^{1-2^k} \\ b^{(k)}(z)&=\frac{2^k-1}{2^k}z \end{aligned} \tag{6.2b}$$

which is still a closed curve, which under the transformation

$$\lambda^{(k)} = \frac{1}{2^k} \exp i\Lambda^{(k)}$$

$$b^{(k)} = \frac{2^k - -1}{2^k} \exp iB^{(k)}$$

becomes

$$\Lambda^{(k)}(\theta) = -(2^k - 1)\theta$$

$$B^{(k)}(\theta) = \theta$$

where the domains of the parameters are the usual ones.

It is evident that Proposition 5 can be restated in this case as follows.

Proposition 5′. We have

$$\begin{aligned} \lim_{k \to +\infty} \sup_{z \in J} |\lambda^{(k)}(z)| &= \lim_{k \to +\infty} \frac{1}{2^k} = 0 \\ \lim_{k \to +\infty} b^{(k)}(z) &= \lim_{k \to +\infty} \frac{2^k - 1}{2^k} z = z \end{aligned} \tag{6.3}$$

so that the limit envelope is again a copy of the Julia set lying in the plane $\lambda = 0$.

In the case $p = 2$ the Julia set $J = [-2, 2]$; this is a singular case since $z = 0 \in J$, where $T'(0) = 0$, so that the envelope of $T(z) = z^2 - 2$,

$$b^{(1)}(\lambda^{(1)}) = \frac{1}{4\lambda^{(1)}} + 2\lambda^{(1)}, \qquad \lambda^{(1)} \in (-\infty, -0.25] \cup [0.25, +\infty)$$

extends to infinity as well as those corresponding to the iterates of T. In fact, the set of points such that $(T^{\circ k})'(z) = T'(z)\, T'(Tz) \cdots T'(T^{\circ k-1}z) = 0$ are given by 0 and its preimages up to order $k-1$. Since $z = 0$ is preperiodic of the fixed point $z = q = 2$, they all stay on the Julia set J. As concerns the kth envelope, it is given by 2^k branches which go to infinity as z approaches one of the singular points from the left or the right. This will be true also in the limit, so that we cannot recover the Julila set J in the complex plane $\lambda = 0$ uniformly in x as in all the cases considered previously.

A possible solution to this problem seems to come from the one-point compactification of the complex plane $\Im b = 0$; $\Im \lambda = 0$, since it allows the first envelope to become homeomorphic to the Julia set J; it then seems natural to expect that in this modified framework one could recover all previous results.

7. CONCLUSIONS

From the above discussion there emerges the existence of a simple and natural way of associating an infinite set of linear mappings with untrivial parameters to a Cantor set through a Legendre transformation. This association gives a natural framework in which to treat the problem of linear Cantorian approximations (see Theorems 1 and 2) and it seems worth future investigation to study the problem of a complete dynamical and topological reconstruction through sequences of linear IFSs for which the two theorems above assure convergence of the attractors and the corresponding p-balanced measures. In this framework one would suppose which are the exact nonlinear maps and set up a linear approximation along the lines proposed in this paper.

It should be noticed that it is possible also to consider a true inverse problem starting from a "time" series, constructing the approximated envelopes (with will be piecewise linear) and making all the calculations on these approximated envelopes.

More important to us appears the problem of generalizing the results stated here to nonhyperbolic cases. If we consider the case of quadratic maps, when $p > 2$, then we can perform our calculations in $\mathbb{C}$, while in the limiting case $p = 2$ it is necessary to move to the one-point compactification of the complex plane in order to generalize the results, at least as far as the first envelope is concerned. If $p = 0$, the envelope becomes a closed curve in $\mathbb{C}^2$ and for $p > 0$ and sufficiently small the envelope is still a curve and Proposition 5′ may be restated with obvious modifications so as to hold also in this case. We are interested in investigating the properties of the envelope and its images which are independent of the form of the maps and of the values of the parameters associated to the maps themselves. To this aim we are starting with the case of the quadratic maps, since the structure of the Julia sets is better known in this cases.

APPENDIX. A REMARK ABOUT THE MULTIVARIABLE AND MULTIDIMENSIONAL CASE

In this Appendix we make some remarks on the cases of multidimensional variables and maps. Let us start with a simple case in which the maps depend not only on the first predecessor element of the orbit, but also on the second one in a rather peculiar way:

$$x_n = S(x_{n-1}, x_{n-2}) = T(x_{n-1} + \alpha x_{n-2}) \tag{A.1a}$$

Then, with $x = x_{n-1}$, $y = x_{n-2}$, $z = x + \alpha y$, Eq. (3.1) becomes

$$b^{(1)}(x, y) = S(x, y) - \lambda^{(1)} x - \mu^{(1)} y$$

so that parametrically we get

$$\begin{aligned}
\lambda^{(1)} &= \frac{\partial S}{\partial x}(x, y) = T'(z) \\
\mu^{(1)} &= \frac{\partial S}{\partial y}(x, y) = \alpha T' = \alpha \lambda^{(1)} \\
b^{(1)} &= S(x, y) - \frac{\partial S}{\partial x}(x, y) x - \frac{\partial S}{\partial y}(x, y)\, y = S(x, y) - \lambda^{(1)}(x + \alpha y) \\
&= T(z) - \lambda^{(1)}(z) z = T(z) - T'(z) z
\end{aligned} \tag{A.2}$$

It is clear that formally one can reduce this to a one-dimensional problem for the first envelope, since this one lies in a plane. If the maps T are linear that is obviously true for all iterates, since the envelope degenerates to a single point. In the nonlinear case one can calculate the relations between the lth-order envelope and the first-order one, and the expressions one gets are quite tedious and are omitted.

In the case where the maps depend on the first k predecessors of the sequence

$$\begin{aligned}
x_n &= S(x_{n-1}, \dots, x_{n-k}) \\
&= T(\alpha_{1,1} x_{n-1} + \alpha_{1,2} x_{n-2} + \cdots + \alpha_{1,k} x_{n-k}, \\
&\quad \dots, \alpha_{r,1} x_{n-1} + \alpha_{r,2} x_{n-2} + \cdots + \alpha_{r,k} x_{n-k})
\end{aligned} \tag{A.1b}$$

where some $\alpha_{l,i} \neq 0$ for some i for any $l = 1, \dots, r$ with $r < k$, we can arrive at similar conclusions to that of expression (A.2), in the sense that the first envelope lies in an r-dimensional hyperplane of the $(k+1)$-parameter space.

In the d-dimensional case, we will obtain formally analogous expressions in the case in which the component maps $T_{(i)}$ are such that they depend on the linear combination of variables

$$T_{(i)}(x_{(1)}, \dots, x_{(d)}) = F(\alpha_{1,1} x_{(1)} + \cdots + \alpha_{1,d} x_{(d)}, \dots, \alpha_{r,1} x_{(1)} + \cdots + \alpha_{r,d} x^{(d)}) \tag{A.3}$$

for some $\alpha_{i,j}$ constants and $r < s$; then the ith component of the envelope lies in an r-dimensional hyperplane instead of a d-dimensional one, where notations are kept consistent with those used in previous sections.

In particular, if the maps $T_{(i)}$ are linear in all the variables, the envelope will degenerate, as expected, to a point.

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